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Technique on Solving a Binary Quadratic Diophantine Equation $3x^2 + 5y^2 = 17^{2k+1}$

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Abstract: This paper aims at solving a binary quadratic Diophantine equation involving odd powers of 17 represented by $3x^2 + 5y^2 = 17^{2k+1}$. The method employed to solve the given equation is applying the concept of divisibility.

Keywords: Quadratic Diophantine equation, Binary quadratic equation, Integer solutions, Division algorithm.

Notation: [x] Greatest integer function

Definition: (The Division Algorithm)

Let a be any integer and b a positive integer. Then there exist unique integers q and r such that $a = b * q + r, 0 \le r < b$. Notice that the above equation can be written as $\frac{a}{b} = q + \frac{r}{b}, 0 \le \frac{r}{b} < 1$. Consequently, $q = \left[\frac{a}{b}\right]$ and r=a-b*q=a-b*$\left[\frac{a}{b}\right]$.

INTRODUCTION

Diophantine equations, one of the areas of number theory, occupy a pivotal role in the realm of mathematics and have a wealth of historical significance. It is well-known that Diophantine equations are rich in variety. Particularly, finding integer solutions to homogeneous and non-homogeneous quadratic equations with two or more unknowns [1-11] kindled interest among mathematicians and is still a topic of current research. In this context, while collecting problems on the same, the article presented in [12] was noticed and the authors have used the modulus-amplitude form of complex number. In this paper, an attempt has been made to solve binary quadratic Diophantine equation in [12] in an elegant way. The method employed to solve the given equation is applying the concept of divisibility.

Technical Procedure

The binary quadratic Diophantine equation to be solved is

$$3x^2 + 5y^2 = 17^{2k+1} \quad (1)$$

Taking

$$x^2 = X, y^2 = Y \quad (2)$$

in (1), it is written as

$$3X + 5Y = 17^{2k+1} \quad (3)$$

Rewrite (3) as

$$X = \frac{17^{2k+1}}{3} - \frac{5Y}{3}$$

$$= \frac{17^{2k+1}}{3} - Y - \frac{2Y}{3} \quad (4)$$

Now, observe the following pattern:

$$17 = 3 * 5 + 2 = 3 * \left[\frac{17}{3}\right] + 2,$$

$$17^3 = 3 * 1637 + 2 = 3 * \left[\frac{17^3}{3}\right] + 2,$$

$$17^5 = 3 * 473285 + 2 = 3 * \left[\frac{17^5}{3}\right] + 2,$$

And so on. Thus, by induction, we have

$$17^{2k+1} = 3 * \left[\frac{17^{2k+1}}{3}\right] + 2 \quad (5)$$

Using (5) in (4), we get

$$X = \left[\frac{17^{2k+1}}{3}\right] - Y + \frac{(2-2Y)}{3} \quad (6)$$

Let

$$2 - 2Y = 3t$$

From the above equation we obtain

$$Y = 1 - t - \frac{t}{2}$$

Taking
$t = 2s$

In the above equation, we have

$$Y = 1 - 3s \quad (7)$$

Substituting (7) in (6), it is obtained that

$$X = \left[\frac{17^{2k+1}}{3}\right] - 1 + 5s \quad (8)$$

For any given value of k, it is possible to choose s such that the R.H.S. of (7) and (8) are perfect squares. Taking the square-root, one obtains the corresponding integer values of x, y satisfying (1). A few examples are presented below:

Example 1
$k = 0, s = 0 \Rightarrow x = \pm 2, y = \pm 1.$

Example 2
$k = 1, s = -96 \Rightarrow x = \pm 34, y = \pm 17,$
$s = -320 \Rightarrow x = \pm 6, y = \pm 31.$

Example 3
$k = 2, s = -27840 \Rightarrow x = \pm 578, y = \pm 289,$
$s = -7008 \Rightarrow x = \pm 662, y = \pm 145.$

Example 4
$k = 3, s = -8045856 \Rightarrow x = \pm 9826, y = \pm 4913,$
$s = -16003680 \Rightarrow x = \pm 7534, y = \pm 6929.$

Remark: It is worth mentioning that, for each value of k>0, we are getting two pairs of solutions to (1) where as in [12], only one pair has been presented.

CONCLUSION

In this paper, the non-homogeneous quadratic Diophantine equation with two unknowns given by $3x^2 + 5y^2 = 17^{2k+1}$ has been studied for obtaining its non-zero integer solutions. We have employed the fundamental concept in number theory, namely, the fabulous division algorithm to obtain the required integer solutions for the considered quadratic equation. The above concept can be applied to solve similar equations.

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